

Fermionic T-Duality of $AdS_n \times S^n (\times S^n) \times T^m$ using IIA Supergravity

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Abstract:

We show that the string backgrounds $AdS_2 \times S^2 \times T^6$ and $AdS_d \times S^d \times S^d \times T^{10-3d}$ ($d = 2, 3$) are self-dual under a series of bosonic and fermionic T-dualities. We do this using the fermionic Buscher rules derived by Berkovits and Maldacena, thus working at the level of the supergravity fields. This allows us to explicitly track the behaviour of the RR fields, from which we see that we need T-duality along some torus directions. For the $AdS \times S \times S$ cases, which contain cosets of $D(2, 1; \alpha)^{d-1}$, it is necessary to perform bosonic T-duality along some complexified Killing spinors in one of the spheres.

1 Introduction

An important theme in our understanding of the AdS/CFT correspondence [1] has been that both theories have far more symmetry than initially meets the eye. The largest example of this was the discovery of integrability on both sides of the correspondence, giving an infinite tower of conservation laws [2]. But a particularly important example is that of dual conformal symmetry, which began life as a symmetry of scattering amplitudes of $\mathcal{N} = 4$ SYM [3], acting on the momenta in the same way that ordinary conformal symmetry acts on positions. The AdS/CFT connection between scattering amplitudes and Wilson loops is built on this symmetry, associating each amplitude to a string worldsheet in a dual AdS space. The symmetry extends to the whole superconformal group, and it is believed to be part of the structure of integrability [4, 5].

The existence of dual conformal symmetry was later understood to be a consequence of the self-duality of the string background $AdS_5 \times S^5$ [6, 7], meaning that the total effect of a sequence of bosonic and fermionic T-duality transformations is to return us to exactly the same background. The bosonic dualities along boundary directions of AdS_5 map the momenta of the scattering amplitude into the path bounding the Wilson loop. Fermionic T-duality is a cousin of Bosonic T-duality, acting along a Killing spinor, or in fact necessarily a complexified Killing spinor, rather than a Killing vector. It was introduced by [6, 7]; for reviews see [8, 9] and for other perspectives see [4, 10]. As in the bosonic case, one adds a constrained auxiliary field and then integrates out the original field (i.e. original target space co-ordinate, or fermion) to obtain an

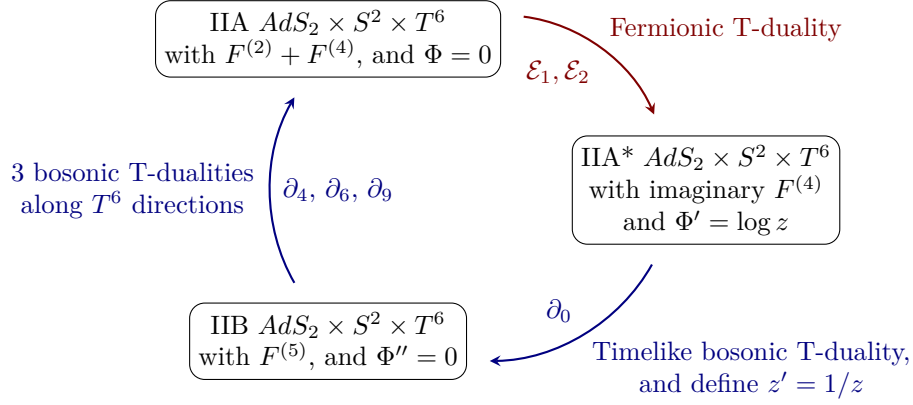


Figure 1: The idea of self-duality we study is that a sequence of fermionic and bosonic T-dualities returns us to the same background. This is depicted here for the case in which we start with IIA $AdS_2 \times S^2 \times T^6$ supported by $F^{(2)}$ and $F^{(4)}$ RR fields (section 3.3).

action of the same shape which can be interpreted as describing strings on a dual background. Rather than performing this process for each background we encounter, we can do it once and write down the rules for mapping one background to another, usually called the Buscher rules [11]. The rules for bosonic T-duality are now understood to be particular generators of the $O(D, D)$ symmetry of generalised geometry [12].

The correspondence between $\mathcal{N} = 4$ SYM and IIB strings in $AdS_5 \times S^5$ is by far the best understood example of AdS/CFT. We are interested in extending this understanding to other examples, with less than maximal supersymmetry. In the correspondence between ABJM theory and IIA strings on $AdS_4 \times CP^3$ [13], dual conformal symmetry is observed in the amplitudes [14] but attempts to show T-self-duality in the string theory have failed [15–18], apart from the much simpler case of the pp-wave limit [19]. We have nothing to add to this puzzle now, but turn instead to AdS_3 and AdS_2 backgrounds.

The study of strings in $AdS_5 \times S^5$ was greatly facilitated by the observation that the Green–Schwarz (GS) action can be written using a \mathbb{Z}_4 -graded coset structure (i.e. the supersymmetric cousin of a symmetric space) $PSU(2, 2|4)/SO(1, 4) \times SO(5)$ [20, 21]. This idea was used heavily in integrability [22], and also in the understanding of fermionic T-duality: Beisert, Ricci, Tseytlin and Wolf [7] work entirely with the coset action, fixing a particular κ -symmetry gauge at the start, and find that the dual model amounts to a different choice of κ -gauge. Similar cosets are useful in describing strings on AdS_4 , AdS_3 and AdS_2 backgrounds, but there are more caveats. The coset $OSP(6|4)/SO(2, 1) \times U(3)$ describes a partially κ -fixed GS string in $AdS_4 \times CP^3$, although this gauge choice forbids some solutions of interest [23].

Fermionic T-duality of $AdS_3 \times S^3 \times T^4$ (with RR flux) was studied using the coset model $PSU(1, 1|2)^2/SU(1, 1) \times SU(2)$ by Adam, Dekel and Oz [15]. This describes GS strings in a κ -symmetry gauge which turns off all the non-coset fermions; like the gauge used to write $AdS_4 \times CP^3$ as a coset, this one is not always legal. Another approach was explored by Ó Colgáin [24], using the Buscher rules derived by Berkovits and Maldacena [6]. This avoids all mention of both cosets and κ -symmetry, working only with the supergravity fields of the background. It also allows one to see the effect on the RR fields explicitly. (This was used for $AdS_5 \times S^5$ in [6]; see also [25, 18] for other applications.)

In this paper we use this idea of working only with the supergravity fields to prove the self-duality of backgrounds $AdS_2 \times S^2 \times T^6$ and $AdS_d \times S_+^d \times S_-^d \times T^{10-3d}$ for $d = 2, 3$. We find it more convenient to begin in the type IIA theory, although the bosonic T-dualities necessarily

take us through IIB. (See figure 1.) For the cases with $S_+^d \times S_-^d$ we show that it is necessary to perform bosonic T-duality along some complexified Killing vectors of one of the spheres.

There is some overlap between this paper and another longer paper [26], written as part of a larger collaboration. The presentation of the supergravity approach there is however very different, using the currents of the coset model to generate Killing spinors, thus treating many backgrounds at once. The aim of the present paper is to give a much more elementary (if less general) explanation of the duality of these backgrounds.

Outline

In section 2 we fix some notation and recall the Buscher rules for fermionic T-duality. Section 3 treats backgrounds $AdS_2 \times S^2 \times T^6$ (for two different choices of RR flux) and $AdS_2 \times S^2 \times S^2 \times T^4$ (which requires bosonic T-duality in a complex direction). Section 4 repeats this one dimension up: $AdS_3 \times S^3 \times S^3 \times S^1$. We conclude in section 5.

Appendix A reviews the necessary facts about bosonic T-duality (including a nontrivial dilaton). Appendix B has gamma matrix conventions.

2 Preliminaries

In type IIA string theory we have two Majorana–Weyl spinors of opposite chirality. Since we use a basis in which $\Gamma_{11} = \sigma^3 \otimes 1$, we can write these as

$$\begin{pmatrix} \epsilon^\beta \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \hat{\epsilon}_\beta \end{pmatrix}, \quad \beta = 1 \dots 16.$$

It is convenient to write the Killing spinors using one 32-dimensional $E = (\epsilon, \hat{\epsilon})$. Our notation is that $\mu, \nu = t, x, z, \theta_+ \dots \psi_-, u$ are curved and $m, n = 0, 1, 2 \dots 9$ flat spacetime indices, and α, β spinor indices as above. The constraints on a Killing spinor E from the gravitino and dilatino variations are

$$D_\mu E = -\frac{1}{8} S \Gamma_\mu E, \quad T E = 0 \quad (1)$$

where the covariant derivative is $D_\mu E = \partial_\mu E + \frac{1}{4} \omega_{\mu np} \Gamma^{np} E$, and we define

$$S \equiv \not{F}^{(2)} \Gamma_{11} + \not{F}^{(4)}, \quad T \equiv \frac{i}{16} \Gamma_m S \Gamma^m = \frac{3i}{8} \not{F}^{(2)} \Gamma_{11} + \frac{i}{8} \not{F}^{(4)}.$$

Fermionic T-duality acts on the dilaton and the RR forms, but leaves the metric and the NS-NS two-form invariant. We will employ here the Buscher rules for this derived by Berkovits and Maldacena [6]. The direction along which we dualise is specified by a complex Killing spinor, which we write $\mathcal{E} = E_1 + iE_2$ etc. We can perform the duality along several directions \mathcal{E}_i at once. These must obey an orthogonality condition

$$\bar{\mathcal{E}}_i \Gamma_\mu \mathcal{E}_j = 0 \quad \forall \mu, i, j, \quad \text{where } \bar{\mathcal{E}} \equiv \mathcal{E}^T \Gamma^0 \quad (2)$$

which is not solved by any real spinors—this is why we must use complex Killing spinors. Then we find a matrix C from

$$\partial_\mu C_{ij} = i \bar{\mathcal{E}}_i \Gamma_\mu \Gamma_{11} \mathcal{E}_j \quad (3)$$

and this gives the change in the dilaton¹

$$\Delta\Phi \equiv \Phi' - \Phi = \frac{1}{2} \log(\det C) \quad (4)$$

and in the RR forms

$$\Delta F \equiv e^{\Phi'} F'^{\alpha}_{\beta} - e^{\Phi} F^{\alpha}_{\beta} = \frac{16}{i} (C^{-1})_{ij} \epsilon_i^{\alpha} \hat{\epsilon}_{j\beta}. \quad (5)$$

Here the lower-case spinors are now complex, $\mathcal{E}_i = (\epsilon_i, \hat{\epsilon}_i)$, and F is the bispinor encoding the RR forms, defined

$$F^{\alpha}_{\beta} \equiv \frac{1}{2!} F_{mn}^{(2)} (\gamma^m)^{\alpha\gamma} (\gamma^n)_{\gamma\beta} + \frac{1}{4!} F_{mnpq}^{(4)} (\gamma^m \gamma^n \gamma^p \gamma^q)^{\alpha}_{\beta}. \quad (6)$$

The lower-case gamma matrices are the following 16×16 blocks (full details are in appendix B)

$$\Gamma^m = \begin{bmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{bmatrix}.$$

3 $AdS_2 \times M$ Backgrounds

This section looks at $AdS_2 \times S^2 (\times S^2) \times T$ backgrounds, all having 8 supersymmetries. However we are interested only in those supersymmetries which commute with the ∂AdS translations along which we will perform bosonic T-duality. This restricts us to considering the 4 Poincaré Killing spinors, allowing for 2 complex directions along which to perform fermionic T-duality.

The metric has a parameter $\alpha \in [0, 1]$, such that at $\alpha = 1$ the second sphere becomes flat:

$$ds^2 = ds_{AdS}^2 + \frac{1}{\alpha} ds_{S_+}^2 + \frac{1}{1-\alpha} ds_{S_-}^2 + \sum_i dy_i^2 = \eta_{mn} e_{\mu}^m e_{\nu}^n dx^{\mu} dx^{\nu}. \quad (7)$$

We adopt Poincaré co-ordinates for AdS_2

$$ds_{AdS_2}^2 = \frac{-dt^2 + dz^2}{z^2}, \quad \Rightarrow \quad \omega_{t[01]} = \frac{1}{z}$$

and for the spheres,

$$ds_{S_{\pm}^3}^2 = d\theta_{\pm}^2 + \sin^2 \theta_{\pm} d\varphi_{\pm}^2, \quad \Rightarrow \quad \omega_{\varphi[23]} = -\cos \theta_+, \quad \omega_{\varphi_-[45]} = -\cos \theta_- \quad (8)$$

and similarly for S_-^3 . Note that the spin connection components $\omega_{\mu ab}$ are independent of α (although the vielbeins e_{μ}^m are of course not). We take $\Phi = 0$, the following RR flux:

$$\mathbb{F}^{(4)} = \Gamma^{01} (-\Gamma^{67} + \Gamma^{89}) + \sqrt{\alpha} \Gamma^{23} (\Gamma^{68} + \Gamma^{79}) + \sqrt{1-\alpha} \Gamma^{45} (-\Gamma^{78} + \Gamma^{69}). \quad (9)$$

At $\alpha = 1$, bosonic T-duality along the x^5 direction takes us to the IIB $AdS_2 \times S^2 \times T^6$ case studied by [27], with only $F^{(5)}$. (Section 3.3 below looks at an alternative IIA choice.) Following [28] we can write

$$\mathbb{F}^{(4)} = -4\Gamma^{0167} P_1 (1 - P_2)$$

where

$$P_1 = \frac{1}{2} (1 + \Gamma^{6789}), \quad P_2 = \frac{1}{2} (1 + \sqrt{\alpha} \Gamma^{012378} + \sqrt{1-\alpha} \Gamma^{014568}).$$

¹ Notice that scaling any \mathcal{E}_i by a constant factor changes Φ' by a constant, and has no effect on ΔF .

The Killing spinor equation is $D_\mu E = -\frac{1}{8}\not{F}^{(4)}\Gamma_\mu E$, and we can build up a solution $E = z^{-1/2}R_{S+}R_{S-}\xi$ as follows:

- We look for Poincaré killing spinors, i.e. those independent of the boundary co-ordinates of AdS . Here this means $\partial_t E = 0$, and thus the rest of the $\mu = t$ equation gives us a constraint

$$E = -\Gamma^{067}P_1P_2E.$$

The $\mu = z$ equation then reads $\partial_z E = -1/2z E$, with solution $E \propto z^{-1/2}$.

- When $\mu \in S_+$ we get (using $P_1E = P_2E = E$)

$$D_\mu E = \frac{\sqrt{\alpha}}{2}\Gamma_\mu\Gamma^{2368}E. \quad (10)$$

Lü, Pope and Rahmfeld [29] gave the Killing spinors for S^2 as follows: the equation $D_\mu \epsilon = \pm \frac{i}{2}e_\mu^a\Gamma_a\epsilon$ is solved by $\epsilon = \exp(\pm i\frac{\theta}{2}\Gamma^2)\exp(\frac{\varphi}{2}\Gamma^{23})$. The θ equation is trivial, and the φ equation uses $(i\Gamma^2)^2 = -1$ and $\{i\Gamma^2, \Gamma^{23}\} = 0$. Our equation replaces $i\Gamma^2 \rightarrow \Gamma^{368}$ which still satisfies these conditions. We also have radius $1/\sqrt{\alpha}$ which does not affect the solution:

$$R_{S+} = \exp\left(\frac{\theta_+}{2}\Gamma^{368}\right)\exp\left(\frac{\varphi_+}{2}\Gamma^{23}\right).$$

- When $\mu \in S_-$ we get instead

$$D_\mu E = \frac{\sqrt{1-\alpha}}{2}\Gamma_\mu\Gamma^{4578}E.$$

Notice that $[\Gamma_\mu\Gamma^{4578}, R_{S+}] = [D_\mu, R_{S+}] = 0$, so this is precisely the equation which must be solved by R_{S-} alone.

The complete solution is then

$$E(\xi) = \frac{1}{\sqrt{z}} \exp\left(\frac{\theta_+}{2}\Gamma^{368}\right)\exp\left(\frac{\varphi_+}{2}\Gamma^{23}\right)\exp\left(\frac{\theta_-}{2}\Gamma^{578}\right)\exp\left(\frac{\varphi_-}{2}\Gamma^{45}\right)\xi \quad (11)$$

with constant ξ obeying

$$-\Gamma^{067}\xi = P_1\xi = P_2\xi = \xi \quad (12)$$

(since the $R_{S\pm}$ commute with these conditions). There are exactly 4 such vectors ξ_a , and we may take them to be orthogonal and normalised: $\xi_a \cdot \xi_b = \delta_{ab}$.

Fermionic T-duality needs complex combinations \mathcal{E}_i of these Killing spinors. In section 3.2 we will want to explore the most general possibility, so let us write these as

$$\mathcal{E}_i = b_{ia}E(\xi_a), \quad a = 1 \dots 4.$$

Then the orthogonality condition (2) reads

$$b_{ia}V_{ab}^\mu b_{jb} = 0, \quad V_{ab}^\mu \equiv \bar{E}_a\Gamma^\mu E_b. \quad (13)$$

In the present AdS_2 case it is easy to show (using $\Gamma^{067}E = -E$) that $V^\mu = 0$ except for $\mu = t$, where

$$V_{ab}^t = E_a^T\Gamma^0 z\Gamma^0 E_b = -\xi_a^T\xi_b = -\delta_{ab}.$$

Thus the orthogonality condition on the coefficients b_{ia} is simply $b_{ia}b_{ja} = 0$.

3.1 Case $\alpha = 1$: $AdS_2 \times S^2 \times T^6$

For the simplest case we can proceed almost exactly parallel to [24]'s treatment of IIB $AdS_3 \times S^3 \times T^4$, although it is even simpler as we have only two complexified Killing spinors.

Using $P_2 E = E$ the $\mu \in S_+$ equation (10) above simplifies to $D_\mu E = \frac{1}{2} \Gamma_\mu \Gamma^1 E$, and thus the solution is just

$$E(\xi) = \frac{1}{\sqrt{z}} \exp\left(\frac{\theta}{2} \Gamma^{21}\right) \exp\left(\frac{\varphi}{2} \Gamma^{23}\right) \xi \quad (14)$$

with ξ_a , $a = 1 \dots 4$ still solving (12):²

$$(\xi_a)^\beta = \frac{1}{2} \delta_{2a-\beta-1} + \frac{1}{2} \delta_{2a-\beta+8} + \frac{1}{2} \delta_{2a-\beta+16} - \frac{1}{2} \delta_{2a-\beta+23}.$$

Choosing the complex combinations $\mathcal{E}_i = b_{ia} E(\xi_a)$ specified by

$$b_1 = (1, 0, i, 0), \quad b_2 = (0, 1, 0, i)$$

and integrating (3), we get

$$C = \frac{1}{z} \begin{bmatrix} i \cos \theta_+ - \sin \theta_+ \sin \phi_+ & -\cos \phi_+ \sin \theta_+ \\ -\cos \phi_+ \sin \theta_+ & i \cos \theta_+ + \sin \theta_+ \sin \phi_+ \end{bmatrix}.$$

Taking the determinant, (4) gives the shift in the dilaton to be

$$\Delta \Phi = \Phi' - \Phi = -\log(z). \quad (15)$$

Bosonic T-duality along time (the only boundary direction of AdS_2) un-does this shift, giving $\Phi'' - \Phi' = \log z$. It also changes the metric as follows:

$$ds_{AdS}^2 = \frac{-dt^2 + dz^2}{z^2} \rightarrow -z^2 dt^2 + \frac{dz^2}{z^2} = \frac{-dt^2 + dz'^2}{z'^2} \quad (16)$$

where in the last step we define $z' = 1/z$ to bring the metric back to its original form.

Turning to the RR forms, ΔF from (5) has real terms cancelling the original flux (9), and imaginary terms leading to

$$\begin{aligned} e^{\Phi'} \not{F}^{(2)'} &= i \Gamma^{14} \\ e^{\Phi'} \not{F}^{(4)'} &= -i \Gamma^{0235} - i \Gamma^{1569} + i \Gamma^{1578}. \end{aligned}$$

Then bosonic duality along directions x^0, x^4, x^6 and x^7 returns us to the original RR flux. Notice that we need to dualise along half of the torus directions here, and that the total number of bosonic dualities is four (as it was in $AdS_5 \times S^5$).³

The earlier treatment of $AdS_2 \times S^2$ in [15, 30] studied only the coset σ -model $PSU(1, 1|2)/U(1)^2$, rather than the whole critical string theory. Thus the need for bosonic T-duality along torus directions was not seen there. This approach was extended in [26] to include non-coset fermions, which forced us to include duality along torus directions.

² i.e. each ξ has four nonzero entries at the positions indicated, with $\beta = 1 \dots 32$ by a temporary abuse of notation.

³ This was also seen in [24]'s supergravity treatment $AdS_3 \times S^3 \times T^4$, where it was necessary to dualise along two torus directions and the two boundary directions of AdS_3 .

3.2 Case $\alpha < 1$: $AdS_2 \times S^2 \times S^2 \times T^4$

Using the setup for generic complex combinations $\mathcal{E}_i = b_{ia} E_a$ above, it is fairly easy to prove that no choice of $b_{i,a}$ can lead to $\Delta\Phi = \log z + \text{const}$. Fermionic T-duality thus necessarily produces a shift in the dilaton which depends on some of the sphere co-ordinates θ_\pm, φ_\pm , which bosonic T-duality along time (and flat torus directions) cannot undo.

The way around this problem is to allow bosonic T-duality along a complex Killing vector. To do this we introduce some unusual co-ordinates for S^2 which arise from the form of the coset element used in [26].⁴ There, $S^2 = SU(2)/U(1)$ is parameterised by

$$g = e^{\lambda_+ L_+} e^{-\lambda_3 L_3} \in SU(2), \quad L_\pm = iL_1 \pm L_2, \quad L_n = \frac{1}{2i} \sigma_n \quad (17)$$

with the coset's denominator taken to correspond to the first factor in $\mathfrak{su}(2) = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(2)}$, where

$$\mathfrak{g}_{(0)} = \langle L_+ + L_- \rangle = \langle L_1 \rangle, \quad \mathfrak{g}_{(2)} = \langle L_+ - L_-, L_3 \rangle = \langle L_2, L_3 \rangle.$$

Then writing $J_{(2)}$ for the projection of the current onto $\mathfrak{g}_{(2)}$:

$$J_{(2)} = [g^{-1} dg]_{(2)} = e^{i\lambda_3} d\lambda_+ L_2 + d\lambda_3 L_3$$

the metric is

$$ds_{S^2}^2 = -2 \text{Tr} (J_{(2)} J_{(2)}) = d\lambda_3^2 + e^{2i\lambda_3} d\lambda_+^2. \quad (18)$$

Comparing this to (8), it is easy to show that the new co-ordinates $x^2 = \lambda_3$ and $x^3 = \lambda_+$ are related to the original ones by

$$\begin{aligned} e^{i\lambda_3} &\equiv \cos \theta_+ + i \sin \theta_+ \cos \varphi_+ \\ \lambda_+ &\equiv e^{-i\lambda_3} \sin \theta_+ \sin \varphi_+ = \frac{\sin \varphi_+}{\cot \theta_+ + i \cos \varphi_+}. \end{aligned} \quad (19)$$

Below we will write $e^3 = e_\mu^3 dx^\mu = \sin \theta_+ d\varphi_+$ etc. for the original co-ordinates, and underlined flat indices for the new complex co-ordinates: $x^2 = \lambda_3$, $x^3 = \lambda_+$. Notice that the volume form is preserved by this change: $e^2 \wedge e^3 = e^2 \wedge e^3$.

The original real S^2 was the slice in which $\theta_+, \varphi_+ \in \mathbb{R}$, but now we allow these to be complex. And importantly, we dualise along the λ_+ direction, as well as time (and some of the torus directions y_i). This gives dual metric

$$ds_{AdS}^{\prime 2} = -z^2 dt^2 + \frac{dz^2}{z^2}, \quad ds_{S^2}^{\prime 2} = d\lambda_3^2 + e^{-2i\lambda_3} d\lambda_+^2 \quad (20)$$

and a shift in the dilaton of

$$\Phi'' - \Phi' = \log (z e^{-i\lambda_3}) = \log \frac{z}{\cos \theta_+ + i \sin \theta_+ \sin \varphi_+}. \quad (21)$$

The dual metric $ds_{S^2}^{\prime 2}$ in terms of θ_+, φ_+ is no longer real. To recover a sphere we must afterwards pick a different real slice: define θ', φ' by (19) with λ_3 replaced by $-\lambda_3$, i.e.

$$e^{-i\lambda_3} = \cos \theta' + i \sin \theta' \cos \varphi', \quad \lambda_+ = \frac{\sin \varphi'}{\cot \theta' + i \cos \varphi'}.$$

⁴ We suppress all except this $SU(2)$ part. We have also re-scaled the algebra to avoid factors $c = \sqrt{\alpha}$. See also [30] for discussion of some related parameterisations.

Then obviously $ds'^2_{S_+} = d\theta'^2 + \sin^2 \theta' d\varphi'^2$. And to recover AdS_2 we must also invert the radial co-ordinate, defining $z' = 1/z$, which clearly gives $ds'^2_{AdS} = (-dt^2 + dz'^2)/z'^2$.

Returning to the fermionic T-duality, we now use very simple complex combinations

$$\mathcal{E}_1 = E_1 + iE_4, \quad \mathcal{E}_2 = -E_2 + iE_3.$$

For this to be meaningful we must specify the eigenvectors ξ_a used; they are easy to calculate but messy to type. Let us order them by the position of the first nonzero component, and choose the sign such that this is positive:

$$(\xi_a)^\beta = 0 \text{ for } \beta < a, \quad (\xi_a)^\beta > 0 \text{ for } \beta = a \quad (22)$$

for example $\xi_3 = \frac{1}{2\sqrt{2}}(0, 0, \sqrt{1+\sqrt{\alpha}}, 0, 0, \sqrt{1-\sqrt{\alpha}}, 0, 0, \dots)$. Then we get

$$C = -2i \frac{\cos \theta_+ + i \sin \theta_+ \cos \varphi_+}{z} \begin{bmatrix} -e^{-i\varphi_-} \sin \theta_- & \cos \theta_- \\ \cos \theta_- & e^{i\varphi_-} \sin \theta_- \end{bmatrix} = C_+ C_- \quad (23)$$

and hence the change in the dilaton is

$$\Phi' - \Phi = \log \frac{\cos \theta_+ + i \sin \theta_+ \cos \varphi_+}{z} + \log(-2i) \quad (24)$$

cancelling the bosonic shift perfectly (apart from a constant which we could have absorbed).

Next we must look at the change in the RR forms, and we begin with the 2-form:

$$\begin{aligned} e^{i\Phi'} F^{(2)'} &= \sqrt{1-\alpha} \frac{(\sin \theta_+ - i \cos \theta_+ \cos \varphi_+) e^1 \wedge e^2 + i \sin \varphi_+ e^1 \wedge e^3}{\cos \theta_+ + i \sin \theta_+ \cos \varphi_+} \\ &= -i\sqrt{1-\alpha} e^1 \wedge e^2. \end{aligned} \quad (25)$$

On the second line we see that this simplifies greatly in terms of the new complex co-ordinates (or rather, the corresponding flat directions). The change in the 4-form field $\Delta F^{(4)}$ has real terms precisely cancelling the original $F^{(4)}$, and imaginary terms which likewise simplify greatly in the new co-ordinates, to give

$$e^{i\Phi'} F^{(4)'} = -i(e^1 e^3 e^6 e^8 + e^1 e^3 e^7 e^9) + i\sqrt{\alpha}(-e^0 e^2 e^6 e^7 + e^0 e^2 e^8 e^9) + i\sqrt{1-\alpha}(e^0 e^3 e^4 e^5). \quad (26)$$

Bosonic T-duality now acts on $F^{(2)'} and $F^{(4)'}$ in the usual way, and we dualise along directions $t = x^0$, $\lambda_+ = x^3$ and x^7, x^8 . This gives $F^{(2)''} = 0$ and$

$$F^{(4)''} = (-e^0 e^1 e^6 e^7 + e^0 e^1 e^8 e^9) + \sqrt{\alpha}(e'^2 e'^3 e^6 e^8 + e'^2 e'^3 e^7 e^9) + \sqrt{1-\alpha}(-e^4 e^5 e^7 e^8 + e^4 e^5 e^6 e^9).$$

We interpret these flat components as being with respect to the new metric ds'^2 (20) created simultaneously by bosonic T-duality, hence $e'^3 = e^{-i\lambda_3} d\lambda_+ \neq e^3$. Note that $e'^2 = -d\lambda_3 = -e^2$, so that the volume form is preserved also by the second change of co-ordinates, to the new real co-ordinates $\theta' = x^{2'}$, $\varphi' = x^{3'}$: we write $e'^2 \wedge e'^3 = e^{2'} \wedge e^{3'}$. Then we have recovered the original flux (9).

3.3 Aside on $AdS_2 \times S^2 \times T^6$ with other RR fluxes

The $\alpha = 1$ background in section 3.1 above has only $F^{(4)}$ turned on. Various other combinations of fluxes are possible with the same geometry, and one of the cases considered in [27] is

$$\mathbb{F}^{(2)} = -\Gamma^{01}, \quad \mathbb{F}^{(4)} = \Gamma^{23}(\Gamma^{45} + \Gamma^{67} + \Gamma^{89}). \quad (27)$$

This is also the case studied in [21], and one of the cases in [26]. It is equally easy to show the self-duality of this background using the same methods, and we look at this briefly before moving on to $AdS_3 \times M$.

We can write

$$S = -4\Gamma^{01}\Gamma_{11}P, \quad T = -4\Gamma^{01}\Gamma_{11}(1 - P)$$

in terms of

$$P = \frac{1}{4}(1 - \Gamma^{6789} - \Gamma^{4589} - \Gamma^{4567}).$$

The Killing spinors are still given by (14), but the constraint on ξ is now

$$\Gamma^0\Gamma_{11}P\xi = \xi.$$

We choose solutions ξ_i which obey (22), and then take the following complex combinations:

$$\mathcal{E}_1 = iE_1 + E_4, \quad \mathcal{E}_2 = iE_2 - E_3.$$

This gives

$$C = \frac{1}{z} \begin{bmatrix} ie^{i\phi_+} \sin \theta_+ & -i \cos \theta_+ \\ -i \cos \theta_+ & -ie^{-i\phi_+} \sin \theta_+ \end{bmatrix}$$

and hence the change in the dilaton is $\Delta\Phi = -\log(z)$. This is the same as (15) above, and is similarly cancelled by the bosonic duality along time.

The change in RR forms is

$$\begin{aligned} \Delta F^{(2)} &= \gamma^{01} \\ \Delta F^{(4)} &= -\gamma^{23}(\gamma^{45} + \gamma^{67} + \gamma^{89}) - i\gamma^{1468} + i\gamma^{1479} + i\gamma^{1569} + i\gamma^{1578}. \end{aligned}$$

Clearly the real terms again cancel the original flux. Acting with bosonic T-duality on the remaining (imaginary) terms along directions $t = x^0$, x^4 , x^6 and x^8 returns us to (27).

4 The background $AdS_3 \times S^3 \times S^3 \times S^1$

The metric here is still of the form (7), but now with one more dimension. We choose nested co-ordinates for the S^3 as in [29]:

$$\begin{aligned} ds_{AdS_3}^2 &= \frac{-dt^2 + dx^2 + dz^2}{z^2} \\ &\Rightarrow \omega_{t[02]} = \omega_{x[21]} = 1/z \\ ds_{S^3}^2 &= d\theta_\pm^2 + \sin^2 \theta_\pm (d\varphi_\pm^2 + \sin^2 \varphi_\pm d\psi_\pm) \\ &\Rightarrow \omega_{\phi_+[34]} = -\cos \theta_+, \quad \omega_{\psi_+[35]} = -\cos \theta_+ \sin \varphi_+, \quad \omega_{\psi_+[45]} = -\cos \varphi_+. \end{aligned}$$

We take the following flux [28]:

$$\begin{aligned} \#^{(4)} &= 2 \left(-\Gamma^{0129} + \sqrt{\alpha} \Gamma^{3459} + \sqrt{1-\alpha} \Gamma^{6789} \right) \\ &= -4\Gamma^{0129}(1-P). \end{aligned} \quad (28)$$

Bosonic T-duality along the x^9 direction returns us to the fluxes of (for instance) [31], and then taking the limit $\alpha \rightarrow 1$ reduces this to the IIB $AdS_3 \times S^3 \times T^4$ case for which [24] studied fermionic T-duality. Because of this study of the $\alpha = 1$ case, we focus here only on the $0 < \alpha < 1$ case.

This background has 16 supersymmetries, and thus 8 Poincaré Killing spinors, allowing for 4 complex directions along which we can perform fermionic T-duality. We can build up a solution $E = z^{-1/2} R_{S+} R_{S-} \xi$ to the Killing spinor equation as before:

- Poincaré killing spinors must now have $\partial_t E = 0$ and $\partial_x E = 0$. The rest of the $\mu = t$ equation gives us a constraint $E = \Gamma^{019} P E$; the $\mu = x$ equation is identical. The $\mu = z$ equation is solved by $E \propto z^{-1/2}$.
- When $\mu \in S_+$ we get (using $P E = E$)

$$D_\mu E = \frac{\sqrt{\alpha}}{2} \Gamma_\mu \Gamma^{01345} E$$

which (again generalising [29]) is solved by a factor $R_{S+} = e^{\frac{\theta_+}{2} \Gamma^{0145}} e^{\frac{\varphi_+}{2} \Gamma^{34}} e^{\frac{\psi_+}{2} \Gamma^{45}}$.

- When $\mu \in S_-$ we get $D_\mu E = \frac{\sqrt{1-\alpha}}{2} \Gamma_\mu \Gamma^{01678} E$. Notice that $[\Gamma_\mu \Gamma^{01678}, R_{S+}] = [D_\mu, R_{S+}] = 0$, so this is precisely the equation which must be solved by R_{S-} alone.

The complete solution is then

$$E(\xi) = \frac{1}{\sqrt{z}} \exp\left(\frac{\theta_+}{2} \Gamma^{0145}\right) \exp\left(\frac{\varphi_+}{2} \Gamma^{34}\right) \exp\left(\frac{\psi_+}{2} \Gamma^{45}\right) \exp\left(\frac{\theta_-}{2} \Gamma^{0178}\right) \exp\left(\frac{\varphi_-}{2} \Gamma^{67}\right) \exp\left(\frac{\psi_-}{2} \Gamma^{78}\right) \xi$$

with constant ξ obeying

$$\Gamma^{019} \xi = P \xi = \xi.$$

There are exactly 8 such spinors, as advertised. Let us order the (orthogonal) ξ_a such that

$$(\xi_a)^\beta = 0 \text{ for } \beta < a, \quad (\xi_a)^\beta = \frac{1}{2} \text{ for } \beta = a.$$

The orthogonality relation is now nontrivial for two values of μ , which in terms of V^μ defined in (13) reads

$$V^t = -1, \quad V^x = -\sqrt{\alpha} \sigma^2 \otimes \sigma^2 \otimes 1_{2 \times 2} - \sqrt{1-\alpha} 1 \otimes \sigma^1 \otimes 1.$$

After some experiments we solve this by taking

$$\mathcal{E}_1 = E_1 - iE_8, \quad \mathcal{E}_2 = E_2 + iE_7, \quad \mathcal{E}_3 = E_3 - iE_6, \quad \mathcal{E}_4 = E_4 + iE_5.$$

With this choice C factorises into

$$C = C_+(z, \theta_+, \varphi_+, \psi_+) C_-(\theta_-, \varphi_-, \psi_-)$$

where $C_+ = 2i \sin \theta_+ (\cos \varphi_+ + i \sin \varphi_+ \cos \psi_+)/z$ is a number, and C_- a matrix with $\det C_- = 1$:

$$C_- = \begin{bmatrix} -c_\theta - i s_\theta s_\varphi (\hat{\alpha} c_\psi + \sqrt{\alpha} s_\psi) & -i \hat{\alpha} c_\varphi s_\theta & -\sqrt{\alpha} c_\varphi s_\theta & s_\theta s_\varphi (\sqrt{\alpha} c_\psi - \hat{\alpha} s_\psi) \\ -i \hat{\alpha} c_\varphi s_\theta & i s_\theta s_\varphi (\hat{\alpha} c_\psi - \sqrt{\alpha} s_\psi) - c_\theta & s_\theta s_\varphi (\sqrt{\alpha} c_\psi + \hat{\alpha} s_\psi) & \sqrt{\alpha} c_\varphi s_\theta \\ -\sqrt{\alpha} c_\varphi s_\theta & s_\theta s_\varphi (\sqrt{\alpha} c_\psi + \hat{\alpha} s_\psi) & c_\theta + i s_\theta s_\varphi (\hat{\alpha} c_\psi - \sqrt{\alpha} s_\psi) & i \hat{\alpha} c_\varphi s_\theta \\ s_\theta s_\varphi (\sqrt{\alpha} c_\psi - \hat{\alpha} s_\psi) & \sqrt{\alpha} c_\varphi s_\theta & i \hat{\alpha} c_\varphi s_\theta & c_\theta - i s_\theta s_\varphi (\hat{\alpha} c_\psi + \sqrt{\alpha} s_\psi) \end{bmatrix}$$

writing (only here) $s_\theta = \sin \theta_-$, $c_\theta = \cos \theta_-$ and $\hat{\alpha} = \sqrt{1 - \alpha}$. Then we get

$$\Phi' - \Phi = 2 \log \frac{2 \sin \theta_+ (\cos \varphi_+ + i \sin \varphi_+ \cos \psi_+)}{z}. \quad (29)$$

To undo this shift using bosonic T-duality we again need to dualise along some complex directions. The parameterisation of the coset element chosen in [26] implies a metric for S_+^3 generalising (18):

$$ds_{S_+^3}^2 = d\lambda_3^2 + e^{2i\lambda_3} d\lambda_+^2 - e^{2i\lambda_3} d\lambda_-^2.$$

Dualising along λ_+ and λ_- (as well as two AdS_3 directions t, x) gives a shift in the dilaton of

$$\Phi'' - \Phi' = 2 \log (z e^{-i\lambda_3})$$

from which we read off

$$e^{i\lambda_3} = \sin \theta_+ (\cos \varphi_+ + i \sin \varphi_+ \cos \psi_+).$$

The same duality also changes the metric $ds^2 \rightarrow d\lambda_3^2 + e^{-2i\lambda_3} d\lambda_+^2 - e^{+2i\lambda_3} d\lambda_-^2$, which as before we can absorb into the relation between λ_3 and the real co-ordinates for S^3 , writing $e^{-i\lambda_3} = \sin \theta' (\cos \varphi' + i \sin \varphi' \cos \psi')$.

Finally we look at the change in the RR fields. Fermionic T-duality produces a 2-form field which, as in (25) above, is much simpler written in the new complex co-ordinates. Writing $e^3 = d\lambda_3$, this is

$$\begin{aligned} \Delta F^{(4)} &= 2\sqrt{1-\alpha} \left\{ \cot \theta_+ e^2 e^3 + \frac{i(\cos \varphi_+ \cos \psi_+ + i \sin \varphi_+) e^2 e^4 - i \sin \psi_+ e^2 e^5}{\sin \theta_+ (\cos \varphi_+ + i \sin \varphi_+ \cos \psi_+)} \right\} \\ &= 2i\sqrt{1-\alpha} e^2 e^3. \end{aligned} \quad (30)$$

One of the $\sqrt{\alpha}$ terms in the change in the 4-form field is similarly simple, on the first line here:

$$\begin{aligned} \Delta F^{(4)} &= -2\sqrt{\alpha} e^0 e^1 e^9 \left[\cot \theta_+ e^3 + \frac{i(\cos \varphi_+ \cos \psi_+ + i \sin \varphi_+) e^4 - i \sin \psi_+ e^5}{\sin \theta_+ (\cos \varphi_+ + i \sin \varphi_+ \cos \psi_+)} \right] \\ &\quad + 2\sqrt{\alpha} e^3 e^4 e^5 e^9 + 2\sqrt{1-\alpha} e^6 e^7 e^8 e^9 - 2e^0 e^1 e^2 e^9 \\ &\quad + 2e^2 e^9 \left[\cot \theta_+ e^4 e^5 + \frac{-i(\cos \varphi_+ \cos \psi_+ + i \sin \varphi_+) e^3 e^5 - i \sin \psi_+ e^3 e^4}{\sin \theta_+ (\cos \varphi_+ + i \sin \varphi_+ \cos \psi_+)} \right]. \end{aligned}$$

On the second line we have terms cancelling the original $F^{(4)}$. The square bracket on the third line here is precisely $i e^4 e^5$, as can be seen by noting that the volume form should be the same as for the initial co-ordinates — the product of the two square brackets above is

$$[i e^3] \wedge [i e^4 \wedge e^5] = (\cot^2 \theta_+ + \dots) e^3 \wedge e^4 \wedge e^5 = -e^3 \wedge e^4 \wedge e^5.$$

Thus the RR fields after the fermionic step are:

$$\begin{aligned} e^{\Phi'} F^{(2)'} &= 2i\sqrt{1-\alpha} e^2 e^3 \\ e^{\Phi'} F^{(4)'} &= 2i\sqrt{\alpha} e^0 e^1 e^3 e^9 + 2i e^2 e^4 e^5 e^9. \end{aligned} \tag{31}$$

Bosonic T-duality along t , x , and λ_{\pm} (i.e. directions 0,1,4,5) leads to

$$e^{\Phi''} F^{(4)''} = -2e^0 e^1 e^2 e^9 + 2\sqrt{\alpha} e'^3 e'^4 e'^5 e^9 + 2\sqrt{1-\alpha} e^6 e^7 e^8 e^9.$$

Then using $e'^3 e'^4 e'^5 = e^{3'} e^{4'} e^{5'}$ to write this in terms of the final set of real co-ordinates θ', φ', ψ' , we recover the original RR field, (28).

5 Conclusions

In this paper we have shown that type IIA string theory on backgrounds $AdS_2 \times S^2 \times T^6$ and $AdS_d \times S_+^d \times S_-^d \times T^{10-3d}$ for $d = 2, 3$ is self-dual under bosonic and fermionic T-duality. We have deliberately kept our computations as ‘low-tech’ as possible, constructing Killing spinors by hand following [29], and then using the Buscher rules of [6] to work out the effect of fermionic T-duality. This approach makes particularly clear that we cannot avoid bosonic T-duality along torus directions, and in the $S_+ \times S_-$ cases also along some complexified sphere directions.

All of these backgrounds have also been studied (in various ways) using coset sigma models. These approaches have different strengths and weaknesses:

- $AdS_3 \times S^3$ is the bosonic part of $PSU(1, 1|2)^2/SU(1, 1) \times SU(2)$, which in turn is a truncation of the $PSU(2, 2|4)$ super-coset used for $AdS_5 \times S^5$. This truncation lets you re-use many parts of the story, however it omits the flat torus directions, and half the fermions of the GS action.⁵ It is possible to fix a kappa-gauge such that the non-coset fermions all vanish, as was done by [15]. The approach of [26] was to start with this coset, and then explicitly add the non-coset fermions. This made visible the need for T-duality along torus directions.
- $AdS_3 \times S^3 \times S^3$ is likewise the bosonic part of the exceptional group coset $D(2, 1; \alpha)^2/SO(1, 2) \times SO(3)^2$, which again omits half the GS fermions.⁶ For this case the approach of [26] was different: since all the Killing spinors lie in the coset directions, one can re-cast the fermionic Buscher rules into this language. Once this is done, self-duality follows in a few lines.
- In the case $AdS_2 \times S^2 \subset PSU(1, 1|2)/SO(1, 1) \times U(1)$, only 8 of the GS string’s 32 fermions are described by these coset, thus a kappa gauge cannot remove all the non-coset fermions. (This is also true for $AdS_2 \times S^2 \times S^2 \subset D(2, 1; \alpha)/SO(1, 1) \times SO(2)^2$.) The same list of approaches is possible here: one can truncate to the coset [15], or explicitly add non-coset fermions to the action [26], or generate Killing spinors from the coset to use in the supergravity rules [26].

Apart from these backgrounds, in the introduction we mentioned the goal of resolving the longstanding problem with $AdS_4 \times CP^3$ self-duality [15, 16, 18]. Another open direction concerns backgrounds with mixed RR and NS-NS fields. All of the above geometries can be supported by some such mixture, and some can be supported by pure NS-NS flux [28]. The integrability of

⁵ In the integrable picture, the same truncation of $AdS_5 \times S^5$ omits the massless modes, from the torus directions and their superpartners. This was the first sector to be explored, see review [32]. Many details of the massless sector are still being worked out [33].

⁶ However in this case the coset only omits two bosonic directions, see [34].

such backgrounds has been a topic of much recent interest [35] but nothing is known about their fermionic T-duality.

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A Bosonic T-duality

While our focus here is on fermionic T-duality, the self-duality involves also bosonic T-duality. For this the tools are much better developed, and we review briefly what we need of these.

A.1 Generalised metric and dilaton

The most convenient formalism for working out the effect of bosonic T-duality on the metric G , the Kalb–Ramond antisymmetric B field, and the dilaton Φ is to use generalised geometry, where T-duality is generated by an $O(D, D)$ transformation.

The generalised metric M and generalised dilaton d are defined by

$$M = \begin{bmatrix} G - BG^{-1}B & -BG^{-1} \\ G^{-1}B & G^{-1} \end{bmatrix}, \quad e^{-2d} = e^{-2\Phi} \sqrt{\det G}.$$

$O(D, D)$ transformations act $M \rightarrow M' = T^{-1}MT$, and d is invariant. T-duality is implemented by the following matrix, with for instance $v = (0, \dots, 0, 1)$ to dualise along the last direction:

$$T_v = \begin{bmatrix} 1 - n & n \\ n & 1 - n \end{bmatrix}, \quad n_{\mu\nu} = v_{\mu}v_{\nu}.$$

Then it is easy to see that dualising along several co-ordinate directions, we have $T_{v_1}T_{v_2} = T_{v_1+v_2}$. Reading off G'^{-1} from M' , the change in the dilaton is clearly

$$\Delta\Phi = \frac{1}{4} \log \frac{\det G'}{\det G}. \quad (32)$$

When $B = 0$, if we write $g_{\mu\nu}$ for the block of $G_{\mu\nu}$ in the directions along which we are dualising, then this becomes simply

$$\Delta\Phi = -\frac{1}{2} \log(\det g).$$

A.2 Effect on Ramond–Ramond fields

One way to work out the changes in the RR form fields, from Fukuma et al. [36], begins by writing them using a set of fermionic creation operators ψ^m . In terms of the potentials $C^{(p)}$ this reads

$$|C\rangle = \sum_p \frac{1}{p!} C_{mn\dots p}^{(p)} \psi^m \psi^n \dots \psi^p |0\rangle$$

where the algebra is

$$\{\psi_m, \psi^n\} = \delta_m^n, \quad \{\psi^m, \psi^n\} = \{\psi_m, \psi_n\} = 0$$

and destruction operators ψ_m annihilate the vacuum $|0\rangle$. Then the action of bosonic T-duality in the i direction is given by

$$T_m = \psi^m + \psi_m. \quad (33)$$

Note that different T_m clearly anti-commute, so the sign of the final $C^{(p) \prime}$ depends on the order (but is not physical). However there are two additions to this which matter for us:

- If there is a nontrivial dilaton, these rules should apply instead to $e^\Phi dC^{(p)}$. This is mentioned in the very last paragraph of [36] (before the appendix), but is also implicit in [6, 24].
- T-duality along the time direction exchanges real and imaginary fluxes [37], which means that (33) applies to $m \neq 0$, and $T_0 = i(\psi^0 + \psi_0)$.
- Also note that if there is a B -field turned on, then these rules should apply not to C but to the modified potential D introduced by [36]. We do not consider this case.

Another way to write this was given by Hassan [38], acting directly on the fields $F^{(p)}$. When dualising along direction x^9 , the new fluxes are

$$\begin{aligned} F_{9no\dots q}^{(p) \prime} &= F_{no\dots q}^{(p-1)} - \frac{p-1}{G_{99}} G_{9[n} F_{9o\dots q]}^{(p-1)} \\ F_{mn\dots q}^{(p) \prime} &= F_{mn\dots q}^{(p+1)} - p B_{9[n} F_{o\dots q]}^{(p) \prime} \end{aligned} \quad (34)$$

with $m, n, \dots q = 0, 1, \dots 8$, and antisymmetrisation $[n \dots 9, o, \dots q]$ acting on the letters only. In all our cases G is diagonal and B is zero, and thus the second terms here vanish. Note that when dualising along time we still need to insert a factor of i [37]. And with a nontrivial dilaton we should include its factor too.

We can write the transformation very simply in terms of the bispinor defined in (6) above, and its IIB cousin

$$F^{\alpha\beta} = F_m^{(1)} (\gamma^m)^{\alpha\beta} + \frac{1}{3!} F_{mnp}^{(3)} (\gamma^m)^{\alpha\gamma} (\gamma^n)_{\gamma\delta} (\gamma^p)^{\delta\beta} + \frac{1}{2 \times 4!} F_{mnop}^{(5)} (\gamma^m \gamma^n \gamma^o \gamma^p \gamma^q)^{\alpha\beta}. \quad (35)$$

Incorporating e^Φ as above, we then have⁷

$$\begin{aligned} T^m : \quad e^{\Phi'} F'_{\alpha\beta} &= e^\Phi (\gamma^m)_{\alpha\delta} F^\delta_{\beta}, & \text{or} & \quad e^{\Phi'} F'^{\alpha}_{\beta} = e^\Phi (\gamma^m)^{\alpha\delta} F_{\delta\beta}, & m \neq 0 \\ T^0 : \quad e^{\Phi'} F'_{\alpha\beta} &= i e^\Phi (-\delta_{\alpha\delta}) F^\delta_{\beta}, & \text{or} & \quad e^{\Phi'} F'^{\alpha}_{\beta} = i e^\Phi (\delta^{\alpha\delta}) F_{\delta\beta}. \end{aligned} \quad (36)$$

In fact we always need four dualities, and acting on the F' produced by fermionic T-duality this reads

$$e^{\Phi''} F'' = \pm i \gamma^{0mnp} (e^{\Phi'} F')$$

where m, n, p vary in the different backgrounds. (We have always chosen the sign \pm , or equivalently the order of the dualities, to produce the tidiest result.)

⁷ Notice that (33) and (34) appear to produce fields $F^{(p)}$ with $p > 5$. These are of course not independent, and the IIA ones are related by [36]

$$*F^{(8)} = F^{(2)}, \quad *F^{(6)} = -F^{(4)}.$$

This fact is built into the bispinor (6), where it amounts to $\prod_{m=0}^9 \gamma^m = 1$.

B Gamma Matrices

We adopt the same conventions as [9, 24] among others, defining

$$\Gamma^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Gamma^m = \begin{bmatrix} 0 & (\gamma^m)^{\alpha\beta} \\ (\gamma^m)_{\alpha\beta} & 0 \end{bmatrix}$$

with 16×16 blocks as follows:

$$\begin{aligned} \gamma^1 &= \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 & \gamma^6 &= \sigma^2 \otimes \sigma^2 \otimes 1 \otimes \sigma^1 \\ \gamma^2 &= \sigma^2 \otimes 1 \otimes \sigma^1 \otimes \sigma^2 & \gamma^7 &= \sigma^2 \otimes \sigma^2 \otimes 1 \otimes \sigma^3 \\ \gamma^3 &= \sigma^2 \otimes 1 \otimes \sigma^3 \otimes \sigma^2 & \gamma^8 &= \sigma^2 \otimes 1 \otimes 1 \otimes 1 \\ \gamma^4 &= \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes 1 & \gamma^9 &= \sigma^3 \otimes 1 \otimes 1 \otimes 1 \\ \gamma^5 &= \sigma^2 \otimes \sigma^3 \otimes \sigma^2 \otimes 1 \end{aligned}$$

and $(\gamma^0)_{\alpha\beta} = -\delta_{\alpha\beta}$, $(\gamma^0)^{\alpha\beta} = \delta^{\alpha\beta}$. These γ^m are all clearly real and symmetric. We take the charge conjugation matrix to be $C = \Gamma^0$. And we notice that $\Gamma_{11} \equiv \prod_{m=0}^9 \Gamma^m = \sigma^3 \otimes 1_{16 \times 16}$.

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